Topic : Real Numbers **Class : X**

Concepts

1. Euclid's Division Lemma
2. Euclid's Division Algorithm
3. Prime Factorization
4. Fundamental Theorem of Arithmetic
5. Decimal expansion of rational numbers

A dividend can be written as, Dividend = Divisor × Quotient + Remainder. This brings to Euclid's division lemma.

Euclid's division lemma:
Euclid's division lemma, states that for any two positive integers 'a' and 'b' we can find two whole numbers 'q' and 'r' such that $a = b \times q + r$ where $0 \le r < b$.

Euclid's division lemma can be used to find the highest common factor of any two positive integers and to show the common properties of numbers.

The following steps to obtain H.C.F using Euclid's division lemma:

1. Consider two positive integers 'a' and 'b' such that $a > b$. Apply Euclid's division lemma to the given integers 'a' and 'b' to find two whole numbers 'q' and 'r' such that, $a = b \times q + r$.
2. Check the value of 'r'. If $r = 0$ then 'b' is the HCF of the given numbers. If $r \neq 0$, apply Euclid's division lemma to find the new divisor 'b' and remainder 'r'.
3. Continue this process till the remainder becomes zero. In that case the value of the divisor 'b' is the HCF (a , b). Also HCF(a ,b) = HCF(b, r).

Let us look at some more examples.

Example 1:

Find the H.C.F. of 4032 and 262 using Euclid's division algorithm.

Solution:

Step 1:

First, apply Euclid's division lemma on 4032 and 262.

$4032 = 262 \times 15 + 102$

Step 2:

As the remainder is non-zero, we apply Euclid's division lemma on 262 and 102.

$262 = 102 \times 2 + 58$

Step 3:

Apply Euclid's division lemma on 102 and 58.

$102 = 58 \times 1 + 44$

Step 4:

Apply Euclid's division lemma on 58 and 44.

$58 = 44 \times 1 + 14$

Step 5:

Apply Euclid's division lemma on 44 and 14.

$44 = 14 \times 3 + 2$

Step 6:

Apply Euclid's division lemma on 14 and 2.

$14 = 2 \times 7 + 0$

In the problem given above, to obtain 0 as the remainder, the divisor has to be taken as 2. Hence, 2 is the H.C.F. of 4032 and 262.

Note that Euclid's division algorithm can be applied to polynomials also.

Example 2:

A rectangular garden of dimensions 190 m × 60 m is to be divided in square blocks to plant different flowers in each block. Into how many blocks can this garden be divided so that no land is wasted?

Solution:

If we do not want to waste any land, we need to find the largest number that completely divides both 190 and 60 and gives the remainder 0, i.e., the H.C.F. of (190, 60).

To find the H.C.F., let us apply Euclid's algorithm.

$190 = 60 \times 3 + 10$

$60 = 10 \times 6 + 0$

Therefore, the H.C.F. of 190 and 60 is 10.

Therefore, there will be $\frac{190}{10} = 19$ square blocks along the length of the garden and $\frac{60}{10} = 6$ blocks along its breadth.

Hence, the total number of blocks in the garden will be $19 \times 6 = 114$.

Example 3:

Prove that every positive integer is of the form $3p$, $3p + 1$, or $3p + 2$, where p is any integer.

Solution:

Let a be any positive integer and let $b = 3$.

Applying Euclid's algorithm to a and 3:

$a = 3p + r$; for some integer p and $0 \leq r < 3$

Therefore, a can be $3p$, $3p + 1$, or $3p + 2$.

As a is a positive integer, we can say that any positive integer is of the form $3p$, $3p + 1$, or $3p + 2$.

Example 4:

Prove that every positive even integer is of the form $2m$ and every positive odd integer is of the form $2m + 1$, where m is any integer.

Solution:

Let a be any positive integer and let $b = 2$.

According to Euclid's division lemma, there exist two unique integers m and r such that

$a = bm + r = 2m + r$, where $0 \leq r < 2$.

Thus, $r = 0$ or 1

If $r = 0$, i.e., if $a = 2m$, then the expression is divisible by 2. Thus, it is an even number.

If $r = 1$, i.e., if $a = 2m + 1$, then the expression is not divisible by 2. Thus, it is an odd number.

Thus, every positive even integer is of the form $2m$ and every positive odd integer is of the form $2m + 1$.

Fundamental Theorem of Arithmetic:

Fundamental Theorem of Arithmetic states that every composite number greater than 1 can be expressed or factorized as a unique product of prime numbers except in the order of the prime factors.

We can write the prime factorization of a number in the form of powers of its prime factors.

By expressing any two numbers as their prime factors, their highest common factor (HCF) and lowest common multiple (LCM) can be easily calculated.

The HCF of two numbers is equal to the product of the terms containing the least powers of common prime factors of the two numbers.

The LCM of two numbers is equal to the product of the terms containing the greatest powers of all prime factors of the two numbers.

For any two positive integers a and b, HCF(a , b) x LCM(a , b) = a x b.

For any three positive integers a, b and c,

LCM(a, b, c) = a.b.c HCF(a, b, c) HCF(a, b).HCF(b, c).HCF(c, a)

HCF(a, b, c) = a.b.c LCM(a, b, c) LCM(a, b).LCM(b, c).LCM(c, a) .

Rational number:

A number which can be written in the form ${}^{a}/_{b}$ where a and b are integers and $b \neq 0$ is called a rational number.

Rational numbers are of two types depending on whether their decimal form is terminating or recurring.

Irrational number:

A number which cannot be written in the form ${}^{a}/_{b}$, where a and b are integers and b $\neq 0$ is called an irrational number. Irrational numbers which have non-terminating and non-repeating decimal representation.

The sum or difference of a two irrational numbers is also rational or an irrational number.

The sum or difference of a rational and an irrational number is also an irrational number.

Product of a rational and an irrational number is also an irrational number.
Product of a two irrational numbers is also rational or an irrational number.

Show that the expressions given below are composite numbers.

(a) $3 \times 5 \times 7 \times 23 + 2 \times 7 \times 11 \times 13$

(b) $29 \times 35 + 14$

(c) $3^4 + 6^3$

Solution:

(a) $3 \times 5 \times 7 \times 23 + 2 \times 7 \times 11 \times 13 = 7(3 \times 5 \times 23 + 2 \times 11 \times 13)$

$= 7(345 + 286)$

$= 7 \times 631$

Since both 7 and 631 are prime numbers, we have expressed the given expression as the product of two prime numbers. We know that according to the fundamental theorem of arithmetic, every composite number can be uniquely written as the product of its prime factors. Thus, the given expression represents a composite number.

Example 1:

Write the prime factorization of 31250. What are its prime factors?

Solution:

$31250 = 2 \times 15625$

$= 2 \times 5 \times 3125$

$= 2 \times 5 \times 5 \times 625$

$= 2 \times 5 \times 5 \times 5 \times 125$

$= 2 \times 5 \times 5 \times 5 \times 5 \times 25$

$= 2 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$

$= 2 \times 5^6$

Hence, 2×5^6 is the prime factorisation of 31250. Its prime factors are 2 and 5.

Example 2:

Show that the expressions given below are composite numbers.

(a) $3 \times 5 \times 7 \times 23 + 2 \times 7 \times 11 \times 13$

(b) $29 \times 35 + 14$

(c) $3^4 + 6^3$

Solution:

(a) $3 \times 5 \times 7 \times 23 + 2 \times 7 \times 11 \times 13 = 7(3 \times 5 \times 23 + 2 \times 11 \times 13)$

$= 7(345 + 286)$

$= 7 \times 631$

Since both 7 and 631 are prime numbers, we have expressed the given expression as the product of two prime numbers. We know that according to the fundamental theorem of arithmetic, every composite number can be uniquely written as the product of its prime factors. Thus, the given expression represents a composite number.

(b) $29 \times 35 + 14 = 29 \times 5 \times 7 + 2 \times 7$

$= 7(29 \times 5 + 2)$

$= 7 \times 147$

$= 7 \times 3 \times 7 \times 7$

$= 3 \times 7^3$

Since both 3 and 7 are prime numbers, we have expressed the given expression as the product of its prime factors. We know that according to the fundamental theorem of arithmetic, every composite number can be uniquely written as the product of its prime factors. Thus, the given expression represents a composite number.

(c) $3^4 + 6^3 = 3^4 + (2 \times 3)^3$

$= 3^4 + 2^3 \times 3^3$

$= 3^3(3 + 2^3)$

$= 3^3 \times 11$

Since both 3 and 11 are prime numbers, we have expressed the given expression as the product of its prime factors. It is known that according to the fundamental theorem of arithmetic, every composite number can be uniquely written as the product of its prime factors. Thus, the given expression represents a composite number.

Example:

Find the LCM and the HCF of 432 and 676 using the prime factorization method.

Solution:

We can write these numbers as

$432 = 2^4 \times 3^3$

$676 = 2^2 \times 13^2$

To calculate the HCF

We observe that the only common prime factor is 2 and the smallest power of this prime factor is als

Thus, HCF (432, 676) = $2^2 = 4$

To calculate the LCM

We observe that the prime factors of 432 and 676 are 2, 3, and 13. The greatest powers of these fact respectively.

LCM is the product of the greatest power of each prime factor.

Thus, LCM (432, 676) = $2^4 \times 3^3 \times 13^2 = 73008$

<u>Example:</u>

The HCF of 273 and another number is 7, while their LCM is 3003. Find the other number.

Solution:

Let the first number (a) be 273 and the second number be b.

It is given that HCF (a, b) = 7 and LCM (a, b) = 3003.

We know that HCF × LCM = Product of two numbers.

$\Rightarrow$ HCF $(a, b) \times$ LCM $(a, b) = a \times b$

$\Rightarrow 7 \times 3003 = 273 \times b$

$$\Rightarrow b = \frac{7 \times 3003}{273}$$

$\Rightarrow b = 77$

Hence, the other number is 77.

Example:

Prove that the number 9^n, where n is a natural number, cannot end with a zero.

Solution:

Suppose the number 9^n ends with a zero for some value of n.

Since the number ends with zero, it should be divisible by 10.

Now, $10 = 2 \times 5$

Thus, this number should be divisible by 2 and 5 also.

Therefore, the prime factorization of 9^n should contain both the prime numbers 2 and 5.

We have, $9^n = (3^2)^n = 3^{2n}$

$\Rightarrow$ The only prime in the factorization of 9^n is 3.

Thus, by fundamental theorem of arithmetic, there is no other prime in the factorization of 9^n.

Hence, there is no natural number n for which 9^n ends with the digit zero.

Decimal expansion of rational numbers:

Theorem:
Let p be a prime number. If p divides a^2, then p divides a, where a is a positive integer.

Theorem: If p q is a rational number, such that the prime factorisation of q is of the form 2^a5^b, where a and b are positive integers, then the decimal expansion of the rational number p q terminates.

Theorem: If a rational number is a terminating decimal, it can be written in the form p q , where p and q are co prime and the prime factorisation of q is of the form 2^a5^b, where a and b are positive integers.

Theorem: If p q is a rational number such that the prime factorisation of q is not of the form 2^a5^b where a and b are positive integers, then the decimal expansion of the rational number p q does not terminate and is recurring.

Note: The product of the given numbers is equal to the product of their HCF and LCM. This result is true for all positive integers and is often used to find the HCF of two given numbers if their LCM is given and vice versa.

Example:

Prove that $3-\sqrt{5}$ is irrational.

Solution:

Let us assume $3-\sqrt{5}$ is rational. Then, we can write

$3-\sqrt{5}=\frac{a}{b}$,

where a and b are co-prime and $b \neq 0$.

$\Rightarrow \sqrt{5}=3-\frac{a}{b}$

Now, as a and b are integers, $\frac{a}{b}$ is rational or $3-\frac{a}{b}$ is a rational number.

This means that $\sqrt{5}$ is rational.

This is a contradiction as $\sqrt{5}$ is irrational.

Therefore, our assumption that $3-\sqrt{5}$ is rational is wrong.

Hence, $3-\sqrt{5}$ is an irrational number.

Example:

Write the decimal expansion of $\mathbf{\frac{1237}{25}}$ and find if it is terminating or non-terminating and repeating.

Solution:

Here is the long division method to find the decimal expansion of $\frac{1237}{25}$.

```
     49.48
25)1237.00
   100
    237
    225
     120
     100
      200
      200
        0
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Hence, the decimal expansion of $\frac{1237}{25}$ is 49.48. Since the remainder is obtained as zero, the decimal numberis terminating.

Example:

We can find the decimal expansion of rational numbers using long division method.

However, it is possible to check whether the decimal expansion is terminating or non-terminatin carrying out long division also.

Let us start by taking a few rational numbers in the decimal form.

(a)

$$0.5632 = \frac{5632}{10000}$$

On prime factorising the numerator and the denominator, we obtain

$$\frac{5632}{10000} = \frac{2^9 \times 11}{2^4 \times 5^4} = \frac{2^5 \times 11}{5^4}$$

(b)

$$0.275 = \frac{275}{1000}$$

On prime factorizing the numerator and the denominator, we obtain

$$\frac{275}{1000} = \frac{5^2 \times 11}{2^3 \times 5^3} = \frac{11}{2^3 \times 5}$$